Direct waveform inversion via iterative inverse propagation

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SUMMARY

Seismic waves are the most sensitive probe of the Earth's interior we have. With the dense data sets available in exploration, images of subsurface structures can be obtained through processes such as migration. Unfortunately, relating these surface recordings to actual Earth properties is non-trivial. Tomographic techniques use only a small amount of the information contained in the full seismogram and result in relatively low resolution images. Other methods use a larger amount of the seismogram but are based on either linearization of the problem, an expensive statistical search over a limited range of models, or both. We present the development of a new approach to full waveform inversion, i.e., inversion which uses the complete seismogram. This new method, which falls under the general category of inverse scattering, is based on a highly non-linear Fredholm integral equation relating the Earth structure to itself and to the recorded seismograms. An iterative solution to this equation is proposed. The resulting algorithm is numerically intensive but is deterministic, i.e., random searches of model space are not required and no misfit function is needed. Impressive numerical results in 1D are shown for several test cases.

INTRODUCTION

Seismic waves provide the best and most direct probe available of the properties of the Earth's interior. In exploration seismology artificially generated waves recorded at the Earth's surface are commonly used to image material discontinuities in the subsurface (e.g. via migration). However, a more elusive goal is inversion, the determination of Earth properties such as wave velocity and density from seismic data. Tomographic inversions, which use only the traveltimes of isolated arrivals, are frequently used to obtain smooth, low resolution estimates of the wave velocity generally for use in imaging algorithms. Inversion methods which are designed to make use of all the information in the data, i.e., not only the arrival time and amplitude of pulses but the details of their shape as well, fall under the category of "waveform inversion". The additional information contained in the waveforms promises to improve the accuracy and resolution of Earth structure models.

Although elegant in theory, waveform inversion has not enjoyed much use partly because it is computationally expensive—more than an order of magnitude more expensive than migration—and partly because it requires expert user intervention to make it work. However, in more recent years, faster computers, better inversion strategies (e.g. Pratt, 1999a, 1999b; Sirgue and Pratt, 2004; Yokota and Matsushima, 2004), and a greater demand for accurate information on subsurface structure have made this approach more and more appealing.

Most of the work done to date in waveform inversion falls into one or both of two broad categories: A) linearized methods and B) model-searching methods. Model-searching methods (see e.g. Sen and Stoffa, 1995) involve three main components: 1) the definition of a misfit or error function that measures the difference between seismic waveform data and synthetic seismograms for a proposed Earth model; 2) a pre-defined set of adjustable model parameters and parameter ranges; and 3) some rule or set of rules for searching the defined parameter space. The general idea is to find the combination of model parameters that minimizes the error. This sort of approach can work well if the number of possible parameter combinations to be examined is small. Otherwise, given that the generation of synthetic seismograms is usually quite expensive computationally in three spatial dimensions, searching a large-dimensional model space is impractical at this time.

Linearized methods are, on the other hand, quite cheap computationally.

A mainstay of this approach is to assume that the true, unknown Earth structure differs only slightly from some initial reference model. Under this assumption, one can assume that the true wavefield can be approximated by a combination of the wavefield in the reference model and a singly-scattered wavefield. This single-scattering assumption is known as the first-order Born approximation. Mathematically, this is written as

$$\psi(\mathbf{x}, \omega) \approx \psi_0(\mathbf{x}, \omega) + \omega^2 \int d^3\mathbf{x}' G_0(\mathbf{x}, \omega; \mathbf{x}') V(\mathbf{x}') \psi_0(\mathbf{x}', \omega), \quad (1)$$

where ψ and ψ_0 are the actual and reference wavefields, respectively; G_0 is the Green's function of the reference medium; V is the difference between the reference and actual media (and is frequently known as the "scattering potential"); and ω is the angular frequency. Because this approximation is linear in V , one can construct algorithms to extract V and, hence, an approximation to the true model from measurements of ψ on the Earth's surface. Of course, there are many practical considerations which can make this process difficult to perform, not the least of which is insufficient data. However, even from a theoretical perspective, there are problems. Specifically, if the true structure differs more than a little from the reference model—the usual case—the data can contain arrivals, most notably multiples, that cannot be predicted from the single scattering approximation.

One approach to waveform inversion that does not fall into either the linearized or model-searching categories is what is known as inverse scattering. The name, "inverse scattering", comes from the knowledge that wavefields can be viewed as resulting from multiple scattering of wave energy within the Earth. The object is to find some way to invert these multiple scatterings to retrieve the Earth structure. The principal work done in this area uses an approach called the "inverse scattering series".

Originally developed by Jost and Kohn (1952) in quantum physics and later by Moses (1956), the properties of the inverse scattering series have been studied by Prosser (1980), among others. Razavy (1975) was first to apply this method to the seismic inverse problem. In recent years, it has been most extensively explored by Weglein and collaborators (e.g. Weglein et al., 2003). The main idea of this approach is first to write the wavefield as measured on the Earth's surface as a Born series, the multiple-scattering generalization of eq. (1). In a compact symbolic notation, this is written as

$$(\psi)_S = (\psi_0)_S + (G_0 V \psi_0)_S + (G_0 V G_0 V \psi_0)_S + \dots, \quad (2)$$

where the subscript indicates that we evaluate the wavefield only where we can actually measure it. In principle all orders of scattering off the unknown structure V are included in this sum. If we also assume that the quantity V can be written as an infinite series,

$$V = V_1 + V_2 + V_3 + \dots \quad (3)$$

where V_n is n -th order in the recorded data, we can insert this series into the Born series and collect terms of common order in the data. The result is an infinite set of equations for the V_n :

$$\begin{aligned} (\psi - \psi_0)_S &= (G_0 V_1 \psi_0)_S \\ 0 &= (G_0 V_2 \psi_0)_S + (G_0 V_1 G_0 V_1 \psi_0)_S \\ &\vdots \end{aligned} \quad (4)$$

These equations are meant to be solved in a cascade fashion, solving first for V_1 then using that result to obtain V_2 and so forth.

Iterative inverse propagation

By studying these equations numerically in one spatial dimension, Carvalho (1992) has found that this approach does not appear to be convergent for an arbitrary contrast between the reference and the actual medium. Following his result, Weglein and collaborators have studied the sub-series approach, in which various terms in the full inverse scattering series of common form are combined into separate sub-series, each of which is postulated to perform a different task corresponding to a conventional processing step (Weglein et al., 2003). Using the sub-series approach, they have been able to achieve some promising results with no obvious problems with divergence.

We have derived an alternative approach to the problem of waveform inversion—one which also falls into the category of inverse scattering. We introduce this new development in the next section.

THEORY

Derivation

To introduce and test the concepts of our approach, we present the theory in one spatial dimension. The data will consist of a single trace recorded at $z = 0$ for a source also located at $z = 0$. The source-time function will be a δ -function, making this single trace the complete impulse response for the chosen source/receiver pair. The unknown structure we will wish to recover is assumed to be located “beneath” the source/receiver, along the half-line $z > 0$. Further constraints we impose on the physics of the problem are that we have purely acoustic propagation, fully variable velocity, no attenuation, and constant density. This last constraint is imposed not only for simplicity but for the reason that it is not possible to reconstruct both velocity *and* density from a single trace in 1D.

We take as our reference, or background, a constant-velocity medium with wave propagation velocity c_0 and associated Green’s function G_0 . Let $c(z)$ and G be the wave velocity and Green’s function of the true medium, respectively. We define the scattering potential as

$$V(z) = \frac{c_0^2}{c^2(z)} - 1. \quad (5)$$

The basis of our derivation will be what is known as the Lippmann-Schwinger equation, the integral equation equivalent to the more frequently seen differential equation for acoustic wave propagation:

$$G(z, \omega; z_0) = G_0(z, \omega; z_0) + \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} dz' G_0(z, \omega; z') V(z') G(z', \omega; z_0). \quad (6)$$

One method of solving this equation results in the generation of the Born series or, at least, the generation of Born approximations of various orders. For instance, the first-order Born approximation (also known simply as *the* Born approximation) is obtained by approximating G by G_0 above on the right only, yielding

$$G(z, \omega; z_0) \approx G_0(z, \omega; z_0) + \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} dz' G_0(z, \omega; z') V(z') G_0(z', \omega; z_0). \quad (7)$$

The second-order Born approximation can be obtained by first replacing G on the right-hand side of eq. (6) with the entirety of the right-hand side,

$$\begin{aligned} G(z, \omega; z_0) &= G_0(z, \omega; z_0) + \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} dz' G_0(z, \omega; z') V(z') G_0(z', \omega; z_0) \\ &+ \frac{\omega^4}{c_0^4} \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dz'' G_0(z, \omega; z') V(z') G_0(z', \omega; z'') V(z'') G(z'', \omega; z_0), \end{aligned} \quad (8)$$

and then setting $G \approx G_0$ on the right again. The general rules for obtaining approximations of arbitrary order are

$$\begin{aligned} G^{(1)}(z, \omega; z_0) &= G_0(z, \omega; z_0) \\ G^{(n)}(z, \omega; z_0) &= G_0(z, \omega; z_0) \\ &+ \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} dz' G_0(z, \omega; z') V(z') G^{(n-1)}(z', \omega; z_0), \end{aligned} \quad (9)$$

where $G^{(n)}$ indicates the n -th approximant to G . For our purposes, however, the once-iterated equation *without* the substitution of G_0 for G , eq. (8), is what we want.

To greatly shorten some of the equations to follow, we define some helpful notation. Setting $z = z_0 = 0$, we let

$$(\mathbf{G}_0 \mathbf{V} \mathbf{G}_0)_{RS} = \frac{\omega^2}{c_0^2} \int_{-\infty}^{\infty} dz' G_0(0, \omega; z') V(z') G_0(z', \omega; 0) \quad (10)$$

and

$$\begin{aligned} (\mathbf{G}_0 \mathbf{V} \mathbf{G}_0 \mathbf{V} \mathbf{G}_0)_{RS} &= \\ \frac{\omega^4}{c_0^4} \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dz'' G_0(0, \omega; z') V(z') G_0(z', \omega; z'') V(z'') G(z'', \omega; 0), \end{aligned} \quad (11)$$

where the subscript RS indicates that the quantities are evaluated only for our selected source/receiver geometry. Thus, our once-iterated equations becomes

$$G(0, \omega; 0) = G_0(0, \omega; 0) + (\mathbf{G}_0 \mathbf{V} \mathbf{G}_0)_{RS} + (\mathbf{G}_0 \mathbf{V} \mathbf{G}_0 \mathbf{V} \mathbf{G}_0)_{RS}. \quad (12)$$

Using the explicit form of the Green’s function G_0 ,

$$G_0(z, \omega; z_0) = \frac{ic_0}{2\omega} e^{i\omega|z-z_0|/c_0}, \quad (13)$$

and the fact that $V(z) \equiv 0$ for all $z < 0$, we find

$$\begin{aligned} (\mathbf{G}_0 \mathbf{V} \mathbf{G}_0)_{RS} &= \frac{\omega^2}{c_0^2} \int_0^{\infty} dz \left(\frac{ic_0}{2\omega} \right)^2 V(z) e^{2i\omega|z|/c_0} \\ &= -\frac{1}{4} \sqrt{2\pi} \tilde{V}(2\omega/c_0), \end{aligned} \quad (14)$$

where $\tilde{V}$ is the Fourier transform of V . Inverting this relationship, we find

$$V(z) = -\frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} (\mathbf{G}_0 \mathbf{V} \mathbf{G}_0)_{RS}. \quad (15)$$

(Note that our Fourier transform convention is that of symmetric normalization for the inverse and forward transforms.) Letting $\tilde{D}(\omega) = G(0, \omega; 0) - G_0(0, \omega; 0)$ and

$$U(z) = -\frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} \tilde{D}(\omega), \quad (16)$$

which is tantamount to a constant-velocity depth migration of the data, we apply the above inverse Fourier transform to the once-iterated equation and rearrange to get

$$V(z) = U(z) + \frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} (\mathbf{G}_0 \mathbf{V} \mathbf{G}_0 \mathbf{V} \mathbf{G}_0)_{RS}. \quad (17)$$

Noting that G on the right-hand side depends on V , we see that this equation is an inhomogeneous, *highly* non-linear Fredholm integral equation of the second kind for the scattering potential V . This equation is a novel result and forms the basis for our approach to waveform inversion.

Iterative inverse propagation

An iterative method of solution

There is no pre-existing mathematical apparatus for solving equations like eq. (17). One possible scheme for solving it—one that should eventually be vetted by mathematicians—can be obtained by analogy to the iterative way of solving the Lippmann-Schwinger equation that we discussed earlier (eqs. 9). We propose to construct a sequence of approximants to V by the following rules:

$$V^{(0)}(z) \equiv 0 \quad (18)$$

$$V^{(n)}(z) = U(z) + \frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} \left(\mathbf{G}_0 \mathbf{V}^{(n-1)} \mathbf{G}_0 \mathbf{V}^{(n-1)} \mathbf{G}^{(n-1)} \right)_{RS}, \quad (19)$$

where we generalize the notation of eq. (11) by substituting $V^{(n-1)}$ for V and use $\mathbf{G}^{(n-1)}$ to indicate the Green's function for a medium with potential $V^{(n-1)}(z)$. Going back to the once-iterated Lippmann-Schwinger equation, eq. (12), which holds for any medium, we have

$$\left(\mathbf{G}_0 \mathbf{V}^{(j)} \mathbf{G}_0 \mathbf{V}^{(j)} \mathbf{G}^{(j)} \right)_{RS} = G^{(j)}(0, \omega; 0) - G_0(0, \omega; 0) - \left(\mathbf{G}_0 \mathbf{V}^{(j)} \mathbf{G}_0 \right)_{RS}. \quad (20)$$

Using this relation in our iterative equation, we can achieve a substantial simplification to our second iteration rule above:

$$V^{(n)}(z) = U(z) + V^{(n-1)}(z) + \frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} \left(G^{(n-1)}(0, \omega; 0) - G_0(0, \omega; 0) \right). \quad (21)$$

In parallel with our definition of U , eq. (16), we set

$$U^{(n-1)}(z) = -\frac{4}{\pi c_0} \int_{-\infty}^{\infty} d\omega e^{-2i\omega z/c_0} \left(G^{(n-1)}(0, \omega; 0) - G_0(0, \omega; 0) \right) \quad (22)$$

to get the final version of our iteration rules:

$$\begin{aligned} V^{(0)}(z) &\equiv 0 \\ V^{(n)}(z) &= U(z) + V^{(n-1)}(z) - U^{(n-1)}(z). \end{aligned} \quad (23)$$

The algorithmic interpretation of these iteration rules is simple. At each step, the new approximant $V^{(n)}$ is obtained from the previous one, the “migrated” data U , and the migrated synthetic data from a medium with scattering potential $V^{(n-1)}$. Thus our algorithm should recover V through a deterministic sequence of forward modelling simulations. It is for this reason that we call our approach to inverse scattering “iterative inverse propagation”.

NUMERICAL EXAMPLES

As proof of concept, we present three synthetic tests of 1D iterative inverse propagation. In each case, we constructed a test model and generated synthetic seismograms with a staggered-grid finite-difference (FD) code (see e.g. Virieux, 1986) using a Gaussian source wavelet with a half-width of about 0.5 s. In the inversion, we used an initial constant background velocity of 1 km/s, and the necessary forward modelling was done with the same FD code and source wavelet that was used to generate the synthetic data. We emphasize that *no* pre-processing, such as removal of multiples, was performed on the data.

For numerical reasons, one deviation from the derived iteration rules, eq. (23), was used. At each iteration, the model obtained from the previous step was used as the new “mapping velocity”, i.e., the velocity used to map time into space. In other words, instead of using the factor of

Figure 1: One layer of thickness 3 km with a wave velocity 40% above background. Comparison of the true model (in red) and the inversion results (in black) are shown for various iterations indicated in each panel by the parameter N .

Figure 2: A smoothed random model with velocity variations of approximately $\pm 40\%$ about background. Note that the inversion recovers even the gradient at the end of the model.

Iterative inverse propagation

Figure 3: One low-velocity layer of thickness 3 km with a wave velocity 40% below background. Note the progressive collapse of multiples into the main structure.

$2z/c_0$ in the exponential of eq. (23), a more general function $T(z)$ —the traveltimes from the source to the point z and back to the receiver—was computed from the previous iteration. This very practical *ad hoc* feature was used to eliminate some instabilities that were otherwise occurring at the bottom end of the structure of any test attempted. In fact, this approach also accelerated convergence and indicates the need to incorporate non-constant reference models in future work.

Figure 1 shows the results for our first test, a model with a single high-velocity layer imbedded in an otherwise homogeneous medium. The results of the first three iterations and iterations 10, 20, 30, and 40 are shown. Both the amplitude of the velocity jump (40% above reference) and the positions of the top and bottom of the layer are quite well recovered, with the essential features of the true model recovered by iteration 20 and only moderate improvement obtained thereafter. Those deviations from the true model that are present are due to the band-limited wavelet used in both the initial synthetic data and the FD modelling performed in the inversion.

Figure 2 demonstrates the ability of the algorithm to reconstruct the details of a complicated, smooth model with fluctuations of approximately $\pm 40\%$ about the reference velocity. We see that by iteration 10 the errors in the inversion result are negligible. It is particularly notable that even the smooth gradient at the end of the model is recovered.

Finally, Figure 3 shows the results for a single low-velocity layer of velocity 40% below reference. The value of this example is that it shows clearly what happens to multiples as the inversion progresses. Specifically, one sees that they are gradually “pushed” upward into the main structure, indicating that the information they provide is instrumental in obtaining an accurate final result.

DISCUSSION

We have shown the potential of iterative inverse propagation to recover the details of wave velocity structures in 1D. There are, of course, going to be many hurdles to overcome in extending this method to practical use. In addition to incorporating non-constant reference media, we must also accommodate irregular source/receiver geometries, elasticity, attenuation, limited frequency ranges, and unknown source wavelets, just to name a few. There will also be the issue of the enormous computational effort necessary to perform the forward wave propagation computations required by the algorithm.

However, it is likely that all of these will be overcome in time, and the straightforward nature of a deterministic algorithm such as this implies substantial long-term benefits to seismic exploration.

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