Phase shift migration using orthogonal beamlet transforms

Charles C. Mosher, ChevronTexaco

Ru-Shan Wu, University of California Santa Cruz

Douglas J. Foster, Phillips Petroleum

Summary

The success of wavelet transforms for seismic data compression leads naturally to the idea of “compressing” numerical wavefield propagators. In spite of numerous efforts (i.e. Beylkin, 1992, Dessing and Wapenaar, 1995, Mosher, Foster, and Wu, 1996) application of orthogonal wavelet bases to wavefield propagation has had limited success. In this work, we examine implementations of phase shift migration using orthogonal wavelet transforms as an illustration of the limitations that orthogonality places on wavelet domain propagators. Since wave propagation has a simple representation in the frequency domain, frequency domain wavelet transforms provide a useful framework for studying wave propagation. In particular, we describe phase shift extrapolators for 2-dimensional wavefields that have been Fourier transformed over time and wavelet transformed over space. The wavelet transform over the space axis is implemented in the wavenumber-frequency domain by complex multiplication of low and high pass wavenumber filter functions to form wave packet trees. Space-wavenumber-frequency transforms are usually referred to as ‘beamlet transforms’ (Wu and Chen 2001), and are closely related to Gaussian beams (Hill, 2001, Albetin et al, 2001). The interaction of beamlet transform filter banks and phase shift wavefield extrapolators are simple complex multiplications. Wavefield propagation in the beamlet domain is complicated, however, by the digital implementation of decimation and upsampling operators used in orthogonal wavelet transforms. Unlike the filter functions, which can be viewed as diagonal matrix operators, the decimation and upsampling operators have significant off-diagonal terms. Since these operators do not commute with the filter and phase shift operators, the effects of the off-diagonal terms must be accounted for in the application of wave propagation operators. Use of filters designed for simple shape and compact support in the wavenumber domain reduces the domain of the interactions, resulting in implementations of phase shift extrapolators that have computational complexity comparable to traditional Fourier approaches. Compact support in the wavenumber domain, however, corresponds to poor localization in the space domain. Use of orthogonal wavelet bases for beam-based wavefield propagators results in a trade-off between computational complexity when the wavelet transform filters overlap, and poor localization in space when the overlap is limited. These results suggest that non-orthogonal transforms may provide a better domain for wave propagation.

Introduction

A considerable number of efforts are currently underway to study the interactions of beam-based transforms and wave propagation. These efforts are primarily focussed on operator compression (i.e. Beylkin, 1992, Dessing and Wapenaar, 1995), exploitation of compact support in the space-time domain (Wu and McMechan, 1995; Albetin et al, 2001), or on Gaussian beam representations (Hill, 2001). Much of the current efforts are based on non-orthogonal transforms. In this work, we examine the use of orthogonal wavelet transforms and the constraints that orthogonality places on wavefield propagators in the wavelet transform domain. Since both wave propagation and wavelet transforms have simple representations in the frequency domain, frequency domain wavelet transforms provide a useful framework for studying the nature of wave propagation in the wavelet domain. In particular, we examine phase shift migration in the space-wavenumber-frequency (beamlet) domain. This work builds on the frequency domain representations of wavelet transforms studied by Mosher and Foster, 1995, and Mosher, Foster, and Wu, 1996, where filter convolutions are replaced by linear filter operators in the Fourier domain.

Figure 1

In this representation, $\mathbf{p}(k, \omega)$ is the input signal in the wavenumber-frequency domain, $\mathbf{p}_{l,h}(k, \omega)$ are the Fourier transforms of the filter bank outputs, $\mathbf{L}$ and $\mathbf{H}$ are Fourier domain filters, and $\mathbf{D}$ and $\mathbf{U}$ are Fourier representations of decimation and upsampling operators. Decimation and upsampling in space is accomplished in the wavenumber domain with the folding operator:

$$F \equiv \frac{1}{2} [p(k, \omega) + p(k + \pi, \omega)] \quad (1)$$

In digital form, the folding operator takes the form:

Phase shift migration using orthogonal beamlet transforms

$$F_{ij} = \begin{cases} 1, i=j \\ 1, i=n/2+j \\ 0 \text{ otherwise} \end{cases}; i=1,n; j=1,n \quad (2)$$

where n is the Fourier transform length in the space direction. The decimation operator D is composed of the first $n/2$ rows of F . Application of the operator D results in an aliased (folded) Fourier transform that is half the length of the original, representing a space series with odd samples discarded. The upsampling operator is the first $n/2$ columns of F , or the transpose of the decimation operator:

$$U = D^T \quad (3)$$

Application of U results in an unfolded transform that is twice the length of the original, representing a space series with zeros inserted between each input sample. Compositing the decimation and upsampling operators returns the original folding operator in equation 1:

$$F = UD \quad (4)$$

Using the matrix notation described above, forward and inverse wavelet (or beamlet) transforms in the frequency (or wavenumber) domain can be expressed as:

$$W_f \equiv \begin{bmatrix} DH \\ DL \end{bmatrix} \quad (5)$$

$$W_f^{-1} \equiv \begin{bmatrix} HU & L'U \end{bmatrix}, \quad (6)$$

where the filter operators H and L satisfy the wavelet transform orthogonality constraints (Daubchies, 1992; Foster, Mosher and Wu, 1997). The forward transform operates on a vector of length n and produces two vectors of length $n/2$, the inverse transform combines two vectors of length $n/2$ and produces a single vector of length n . Wave packet trees are constructed by repeated application of the forward operator W_f to the decimated output vectors (Wickerhauser, 1991). A standard wavelet basis is constructed by repeated application of the forward operator to the low pass vector only (Vetterli and Herley, 1992; Meyer, 1993). Inverse transforms proceed analogously with repeated application of the inverse operator W_f^{-1} .

Beamlet Phase Shift Migration

After 2D Fourier transformation over space and time, a zero offset seismic wavefield recorded at the surface of the earth can be extrapolated using the phase shift extrapolator:

$$p(k_x, \omega; z=\Delta z) = E(k_x, \omega)p(k_x, \omega; z=0) \quad (7)$$

$$E(k_x, \omega) = \exp(\pm i)\Delta z \sqrt{\omega^2/c^2 - k_x^2} \quad (8)$$

where c is propagation velocity (Stolt and Benson, 1986, p. 90).

In the beamlet framework, we perform extrapolation by applying the digital operator E to wavenumber column vectors $p(k_x)$ at each discrete frequency:

$$p' = Ep \quad (9)$$

Where p' is the extrapolated wavefield at $z = \Delta z$. If we define

$$\hat{p} = W_f p = \begin{bmatrix} \hat{p}_h \\ \hat{p}_l \end{bmatrix} = \begin{bmatrix} DHp \\ DLp \end{bmatrix} \quad (10)$$

extrapolation in the beamlet domain can be examined by transforming Equation 9:

$$\begin{aligned} W_f\{p'\} &= W_f\{Ep\} \\ \hat{p}' &= W_f \left\{ E W_f^{-1} W_f p \right\} \\ \hat{p}' &= W_f E W_f^{-1} \hat{p} \end{aligned} \quad (11)$$

We see that $W_f E W_f^{-1}$ is the wave extrapolator in the beamlet domain. In terms of the filter and decimation operators:

$$\hat{E} = W_f E W_f^{-1} = \begin{bmatrix} DHEHU & DHEL'U \\ DLEHU & DLEL'U \end{bmatrix} = \begin{bmatrix} \hat{E}_{hh} & \hat{E}_{hl} \\ \hat{E}_{lh} & \hat{E}_{ll} \end{bmatrix} \quad (12)$$

$$\hat{p}' = \begin{bmatrix} \hat{p}'_h \\ \hat{p}'_l \end{bmatrix} = \begin{bmatrix} \hat{E}_{hh} & \hat{E}_{hl} \\ \hat{E}_{lh} & \hat{E}_{ll} \end{bmatrix} \begin{bmatrix} \hat{p}_h \\ \hat{p}_l \end{bmatrix} \quad (13)$$

Interpretation of the beamlet phase shift extrapolator is complicated by the presence of the cross-coupling operators $\hat{E}_{hl}$ and $\hat{E}_{lh}$. These terms arise from the overlap of the filters L and H in the wavenumber domain. For wavelets with compact support in the space domain, the filters have infinite support in the wavenumber domain. Equation 12 then requires that 4 extrapolations need to be performed. Examination of the nature of the operators in Equation 13 shows that only two extrapolations need to be performed, which are then high and low pass filtered. The result is two times the computational effort of a wavenumber-frequency extrapolation. For more interesting wavelet representations, for example the best-level wavelet basis, the required computations increase by a factor of 2 for every additional level of the transform.

Phase shift migration is implemented in the beamlet domain by repeated application of the phase shift operator for upgoing waves:

Phase shift migration using orthogonal beamlet transforms

$$\begin{bmatrix} \hat{p}_h \\ \hat{p}_l \end{bmatrix}_{z+\Delta z} = \begin{bmatrix} \hat{E}_{hh} & \hat{E}_{hl} \\ \hat{E}_{lh} & \hat{E}_{ll} \end{bmatrix}_{\Delta z} \begin{bmatrix} \hat{p}_h \\ \hat{p}_l \end{bmatrix}_z$$

$$\hat{M}(z+\Delta z) = \begin{bmatrix} \hat{M}_{hh} & \hat{M}_{hl} \\ \hat{M}_{lh} & \hat{M}_{ll} \end{bmatrix} = \sum_{\omega} \begin{bmatrix} \hat{E}_{hh}\hat{p}_h + \hat{E}_{hl}\hat{p}_l \\ \hat{E}_{lh}\hat{p}_h + \hat{E}_{ll}\hat{p}_l \end{bmatrix}_{z+\Delta z} \quad (14)$$

Examples

The domain of interaction for the cross coupling operators is determined by the range of overlap in the filters L and H . For wavelet bases with compact support in the space domain, the filters have infinite support in the wavenumber domain, and there is complete overlap of the filters. An alternative implementation is to use filters with compact support in the wavenumber domain. This makes it possible to limit the filter overlap. In the limiting case, the filters have flat response out to Nyquist, and have no overlap. In the space domain, the convolutional operators are sinc functions with infinite support, and are referred to as Shannon wavelets (Mosher and Foster, 1996).

Figure 2 shows the result of applying a beamlet transform (Equation 5) using Shannon wavelets to a point diffractor. The low pass beamlet retains the low wavenumber energy at the top of the diffractor, and the high pass beamlet retains the limbs. Repeated application of the filters to generate a "level 3" beam packet transform is illustrated in Figure 3a. There are $2^3 = 8$ wave packets with dominant wavenumber increasing from left to right.

Figure 3b shows the result of applying the beamlet migration operator (Equation 8) to the point diffractor. Figure 4 shows the inverse beam packet transform of the migrated point diffractor. For comparison, a conventional phase shift migration of the same point diffractor is shown in Figure 4. As expected, the results are identical.

The complete localization of wavenumbers provided by the Shannon wavelet allows for a simple, efficient implementation of phase shift migration. The space localization is poor, since the corresponding sinc functions decay slowly in the space domain. The choice of an orthogonal wavelet transform limits the flexibility to a trade-off between computational complexity when the filters overlap, and poor space localization when the filters have compact support in the wavenumber domain.

Conclusions

The success of orthogonal wavelet transforms for seismic data compression has motivated a number of efforts to implement seismic wavefield propagation in the "compressed" domain. Analysis of phase shift migration using orthogonal wavelet transforms over the space coordinate (beamlet transforms) shows that orthogonality results in cross-coupling between

wave or beam packets. The effects of cross-coupling can be accounted for, but at the cost of increased computational complexity or poor space localization. The example of phase shift migration shows why non-orthogonal transforms such as Gaussian beam and Gabor-Daubechies decompositions may be a better choice for wave propagation.

References

- Albertin, U., Yingst, D. and Jaramillo, H., 2001, Comparing common-offset Maslov, Gaussian beam, and coherent state migrations, 71st Ann. Internat. Mtg: Soc. of Expl. Geophys., 913-916.
- Beylkin, G., 1992, On the representation of operators in bases of compactly supported wavelets: SIAM Journal of Numerical Analysis, v. 6
- Dessing, J. and Wapenaar, C., 1995, Efficient migration with one-way operators in the wavelet transform domain, 1995 SEG Meeting Abstracts.
- Daubechies, I., 1992, *Ten Lectures on Wavelets*, SIAM
- Foster, D. J., and Mosher, C. C., 1997, Wavelet transforms for seismic data processing, 1997 SEG Meeting Abstracts.
- Hill, N. R., 2001, Prestack Gaussian-beam depth migration: Geophysics, Soc. of Expl. Geophys., 66, 1240-1250.
- Meyer, Y., 1993, *Wavelets: Algorithms and Applications*, SIAM
- Mosher, C. C., and Foster, D. J., 1995, Frequency domain representations of the wavelet transform, Mathematical Methods in Geophysical Imaging III, SPIE Proceedings Series v. 2571
- Mosher, C. C., and Foster, D. J., and Wu, R.S., 1996, Phase shift migration with wave-packet algorithms, Mathematical Methods in Geophysical Imaging IV, SPIE Proceedings Series v. 2822
- Vetterli, M. and Herley, C., 1992, Wavelets and filter banks: theory and design, IEEE Trans. on Signal Proc.
- Wickerhauser, M.V., 1991, Lectures on wavelet packet algorithms, Washington University, Department of Mathematics Technical Report
- Wu, Y., and McMechan, G., 1995, Wave extrapolation in the wavelet domain with application to reverse time migration, 1995 SEG Meeting Abstracts.
- Wu, R.S., and Chen, L., 2001, Beamlet migration using Gabor-Daubechies frame propagator, EAEG 63rd Annual Meeting Abstracts.

Phase shift migration using orthogonal beamlet transforms

Figure 2: Low and high pass beam packets for a point diffractor

Figure 3: (a) Level 3 beam packet transform of a point diffractor (8 beam packets), (b) Migrated point diffractor in the beamlet domain

Figure 4: Beamlet phase shift migration of a point diffractor compared to conventional Fourier phase shift migration